

# Analytical modelling of terminal properties in industrial growth

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In this pedagogical study, carried out by adopting standard mathematical methods of nonlinear dynamics, we have presented some simple analytical models to understand terminal behaviour in industrial growth. This issue has also been addressed from a dynamical systems perspective, with especial emphasis on the concept of the “Balanced Scorecard”. Our study enables us to make the general claim that although the fortunes of an industrial organization can rise with exponential rapidity on relatively short time scales, its growth will ultimately and inevitably be saturated on long time scales by various factors which are nonlinear in character. We have mathematically demonstrated the likely occurrence of this feature under various possible circumstances, including the “Red Ocean” and the “Blue Ocean”. Finally and most importantly, our arguments and their associated mathematical modelling have received remarkable support from the growth pattern indicated by empirical data gathered from a well-recognized global company like *IBM*.

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## I. INTRODUCTION

Dynamic evolution is a ubiquitous attribute of all natural phenomena. This should occasion no surprise, since any physical system that has had a beginning in time, under a given set of initial conditions, has to make its expected passage through time. This is a universal feature, from the very largest scales, as it is with the Universe, whose evolution is described by a plethora of dynamic cosmological models [1], to the very smallest, as it is with the growth of bacterial colonies [2].

This kind of growth through time, however, also has a terminal character about itself. Once again this fact is not very difficult to intuit. On the largest scales the world can only be seen to be of a finite extent, constrained by physical boundaries, and so it could not possibly admit any physical system within itself that will have the advantage of unbridled growth forever. Even if that were to apparently look so in the early stages of the evolution of a system, eventually there would be a saturation towards some terminal limit. This message is driven home in no uncertain terms by palpable facts that mountains can only grow up to a certain maximum height, or stars can remain as a stable physical configuration up to a

certain limiting mass [3].

Mountains and stars, and generally speaking the Universe itself, all evolve much more slowly than events taking place on terrestrial and human time scales, but their seemingly static and timeless appearance is nevertheless overridden ultimately by degeneracy and decay. A realization of this end state of all things though, can be had much more readily in biological systems. A quiet reflection on the extinction of most animal species to have dwelt on the Earth so far, is a sobering reminder of the mortality of all living beings, all of whom have to contend with the rude fact that they have to operate and thrive within a finite space, characterized by parameters of finite size. As a result their environment becomes necessarily competitive, with the resources at their disposal being limited. This makes any indefinite and unconstrained growth an impossibility.

While the dynamic behaviour of any natural phenomenon — be it cosmological, biological, chemical or anything else of a likewise objective nature — can be described by physical laws, enunciated in precise mathematical terms, it is also not difficult to recognize qualitatively similar evolutionary features in systems whose origins lie in the social functionings of the human species. The collective history of mankind is replete with examples of the dramatic rise of civilizations, all of which faded out eventually following a long decline through a twilight period. And so does it happen too to political ideologies, socio-cultural mores and economic principles (with the last of which is connected the fortunes of whole industries).

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In this context we would like to pose the following question : Can the dynamics of a social system be described in equally precise and objective terms as we can do for a natural system? Addressing this question is the principal objective of this work, and to do so we have chosen to study the growth pattern exhibited by an industrial organization. This choice has been inspired by the fact that many aspects of industrial growth lend themselves to reasonably objective analyses. The health of a company is to be seen from the revenue that it is capable of generating, as well as the extent of human resource that it is capable of engaging in achieving its objectives. One could make a quantitative measure of all these variables, and this makes it relatively easy to have a clear understanding of the growth pattern of an industry and posit a mathematical model for it.

Our approach to these issues is predominantly based on the use of standard mathematical tools of nonlinear dynamics and dynamical systems [4, 5]. To explain the growth behaviour of an industrial organization, we have developed our theme by making a pedagogical exposition of the relevance of various mathematical models of increasing complexity. We have found that even when an industrial organization displays conspicuous growth in the early stages, there is saturation of this growth towards a terminal end after the elapse of a certain scale of time. This has been cogently demonstrated in the three following cases:

- Growth in the presence of various inhibitive factors. These factors could be as varied as misalignment within an organization itself, to external factors like competition with a rival, or the diverse socio-economic and ethnic milieu of the environment in which a company might have to operate.
- Growth in which monopolistic control of the theatre of operations has been wrested.
- Growth under conserved and constrained conditions, without any dissipative factors being active.

The conclusion in all the three cases above is the same: One way or the other, as the system size (reflective of the scale of operations) begins to grow through the passage of time, a self-regulatory mechanism takes effective control over growth and gradually drives the system towards a saturated terminal state.

We have subjected our theory to empirical test. We have collected data pertaining to a representative industrial organization, and our preference has been for a company which is global in nature, i.e. its presence is seen and felt all over the world. This choice has been with a design, because we would like to understand the global growth behaviour of a company, whose operating space is by definition on the largest available scale, and, therefore, the overall pattern of its growth would be free of local inhomogeneities. Speaking in terms of analogy, the Universe is also seen to be homogeneous and isotropic on its largest scales [1], and the Earth itself, appearing uneven and of infinite extent on local scales, presents a radically different surface topology when viewed from outer

space. And so it should be that on large scales a clear understanding could also be derived about the constrained feature of the space within which an organization functions.

To this end we have made a study of the cumulative revenue generating capacity and the growth of the human resource strength of the multi-national company, *IBM*. The data obtained were published on the *IBM* website[20] itself. It has been satisfying for us to note that, analyzed according to our objectives and specifications, the *IBM* data actually give a striking match with some of our mathematical models. Both the capacity for revenue generation and the human resource content of *IBM*, over a period of more than ninety years of the existence of the company (this long period actually makes it quite expedient for our study insofar as it is concerned with the growth of an industrial organization from its inception to its terminal stage), show an initial phase of exponential growth, to be followed later by saturated growth towards a terminal end.

Making a final important point is well in order. We have carried out a mathematical analysis of the growth of an industrial organization using methods that are the forte of specialists in nonlinear dynamics. However, the area of application of these analytical tools is, after all, the professional domain of management specialists. The connection between these two vastly diverse disciplines need not be so obvious. So even as we have developed our mathematical models, we have made the necessary and the relevant connection to practical matters of interest for those who are concerned with business administration and the management of industrial organizations. Some of these issues have dwelt on concepts ranging from the “Balanced Scorecard” [6, 7] to the “Red Ocean” and the “Blue Ocean” [8]. We can now safely claim that in all similarly empirical cases where our theoretical methods may be applied, one could come remarkably close to constructing realistic and generic growth models, and that these models would also possess a substantial degree of predictive power.

## II. SATURATED GROWTH IN AN ADVERSE ENVIRONMENT

To appreciate what the recognizable quantitative features of terminal growth behaviour might look like, let us start with a simple power-law model described mathematically by

$$\phi = at^\beta, \quad (1)$$

in which  $\phi$  is the variable whose growth we are interested in (it could be the revenue generating strength or the human resource content of a company),  $t$  is time,  $a$  is a proportionality constant and  $\beta$  is the growth exponent. The first derivative of  $\phi$  with respect to  $t$  is given as

$$\frac{d\phi}{dt} = a\beta t^{\beta-1}. \quad (2)$$

This equation indicates the rate of growth of  $\phi$  through  $t$ . If  $\beta = 1$ , then quite obviously the growth will be perpetually

linear, while if  $\beta > 1$ , then  $\phi$  will display an unbridled and divergent growth through time. For our purposes of understanding terminal growth, however, it will be necessary to consider the case of  $\beta < 1$ . This will mean that with increasing time,  $\phi$  will indeed increase, but this rate of growth itself will be a decreasing function of time. And so as  $t$  increases, the growth of  $\phi$  will continue to slow down.

While all this will be quite true, the model given by Eq. (1) is also overly simplistic. It does convey a basic notion of terminal growth, but it fails to show that with increasing  $t$ , there will be a convergence towards a finite value of  $\phi$ . To achieve this constrained condition, it will be necessary to adopt a slightly more complex model. Let us choose an equation by which the early stages of growth will be conveniently linear (i.e.  $\beta = 1$ ). The slowing down will take place on later time scales. Such a situation can be easily described by the equation,

$$\frac{d\phi}{dt} = a - b\phi, \quad (3)$$

with both  $a$  and  $b$  being positive constants. In physics, this first-order linear ordinary differential equation is easily recognized as bearing the form of the equations which describe Stokes' law of terminal velocity of a particle falling through a viscous medium [9], and the viscoelastic deformation of rocks [10]. Integration of Eq. (3) is trivial, and under the initial condition that  $\phi = 0$  at  $t = 0$ , we will get the analytical solution,

$$\phi = \frac{a}{b} (1 - e^{-bt}). \quad (4)$$

The long-time behaviour of this equation is worth a close look. When  $t \rightarrow \infty$ , there is a convergence of  $\phi$  towards a finite value,  $a/b$ . We could also examine the features of this model on early time scales (quantified by the condition that  $t \ll b^{-1}$ ). In this case,  $e^{-bt} \simeq 1 - bt$ , and, therefore,  $\phi \simeq at$ , very much linear, and very much as we would have wished it to be to demonstrate how rapid growth in the early stages can be slowed down towards a terminal end.

In Fig. 1 we have plotted Eq. (4) for three different values of  $b$ , at a fixed value of  $a$ . Both  $\phi$  and  $t$  have been plotted logarithmically for a better understanding of the power-law behaviour in the early stages (should there be any), and it is easy to see that with a progressive increase in the value of  $b$ , the growth is saturated towards a terminal state at lesser values of  $\phi$ , and sooner in  $t$ .

At this stage it should be quite instructive to study the parameter  $b$  carefully. We can argue that the time scale for the onset of the terminal character is given by  $t \sim b^{-1}$ . Intuitively this is just what it should be. The parameter  $b$  quantifies everything that can retard growth, and so the time scale of terminal behaviour should, qualitatively speaking, be in inverse proportion to it. That this is indeed so is clearly portrayed in Fig. 1. Going back to the retarding term in Eq. (3), it can also be argued that the coupling of  $b$  to  $\phi$  will imply that near the terminal state the growth of  $\phi$  will be limited by its own inflated value.

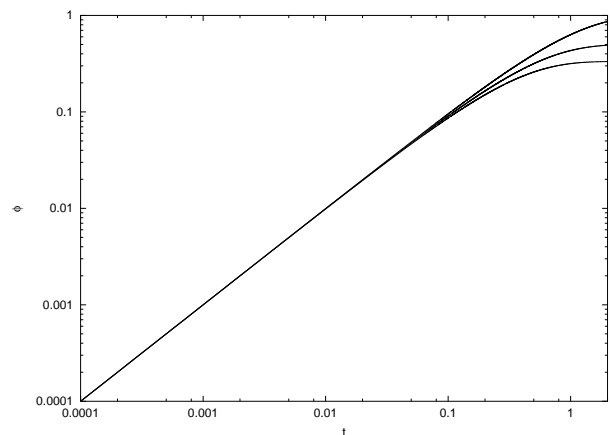


FIG. 1: A logarithmic plot of  $\phi$  against  $t$ , to show how linear growth on small time scales can become saturated towards a terminal value through an infinite passage of time, for a fixed value of  $a$  but for three different values of  $b$ . All three solutions display the same early behaviour (for  $a = 1$ ), but the terminal stage is different for each case, depending on  $b$ . From the top to the bottom the three curves correspond to  $b = 1, 2$  and  $3$ , respectively.

In the context of the growth of an industrial organization, we can identify various factors that may contribute collectively to  $b$ . These factors can be both economic and non-economic in nature. Some may operate internally, while others can make their impact externally. A primary factor is the space within which an organization can be allowed to grow. If this space is constrained to be of a finite size (as, practically speaking, it has to be), then, of course, terminal behaviour becomes a distinct possibility. With its growth, an organization will gradually have to contend with the boundaries of the space within which it has to operate. This brings growth to a slow halt. This can be further aggravated by the presence of rival organizations competing for the same space. The last factor can become particularly acute when a miscalculation is made in assessing future directions of growth vis-a-vis those of rival organizations — both the existing ones and the ones that might emerge in the future.

By way of addressing all this, it might be argued that adopting an active policy based on innovation, diversification or transformation could lead to more sustained growth. However, there are some difficult practical issues to be addressed in this situation too. When an organization transforms itself or diversifies its core competency and objectives, it also runs the risk of rendering a sizeable fraction of its human resources redundant. While this will undoubtedly leave an immediate and tangibly adverse economic impact, the social impact of such measures could be quite considerable too in variously incalculable ways.

Changing political and social values can also be contributory factors in retarding growth. To cite some specific examples in this regard, it is not difficult to perceive that a vigorous governmental pursuit of a policy of disarmament can adversely affect the fortunes of the armaments industry, while a greater public awareness of health has nothing positive to

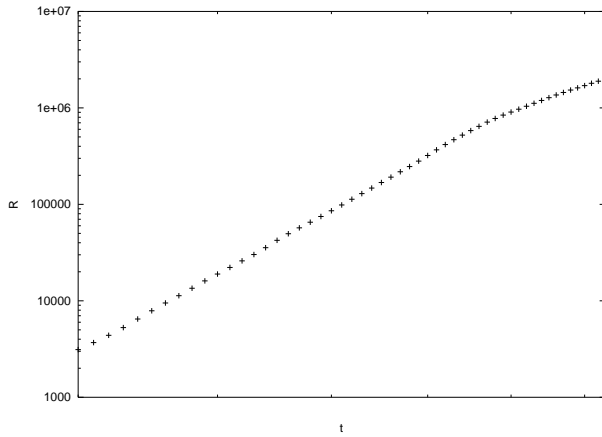


FIG. 2: A logarithmic plot of the cumulative revenue generation capacity,  $R$ , against time,  $t$ . The data show steady growth from the middle of the 1950s till the present date. The later stage of growth shows a distinct swing towards a terminal state.

offer to the tobacco industry.

Increasing the base of operations of an organization by crossing national boundaries may be seen as a lasting solution to problems related to retarded growth. This will open greater opportunities for any organization to operate and grow. While this may look like a self-evident truth, once again, however, such measures will not be free of their associated difficulties. When an organization grows and steps beyond the boundaries of its country of origin, it necessarily tends to stretch its existing resources. Quite naturally this leads to a reduction in the intensity of its control mechanisms, and as a result there arises an early phase of misalignment that dampens the initial spurt in growth. Much conscious effort then needs to be undertaken to ensure functional coherence with the mother company.

Varied socio-cultural barriers across national frontiers are also never too easy to overcome, and adjusting the outlook and objectives of an organization with the native ethnic milieu will, of necessity, involve localizing the original character of the organization. This will ultimately be reflected as localized misalignments in its global functionings, with an attendant dampening of its desired extent of growth. In this situation a significant degree of modification of the policies and procedures of the company will have to be called for, keeping in mind the imperatives of alignment. Complete alignment may be prove to be elusive all the same, and hence, there will be a certain extent of reconciliation to the inevitability of misalignment in operations.

Misalignment could also result from acquisitions and mergers. Achieving an optimal state of consonance in core competency, style of functioning, and objectives between two hitherto different organizations, is almost always fraught with great difficulties. This does nothing to act favourably on the common growth of the newly-merged organizations.

While we have been addressing these issues in general theoretical terms so far, it should now also be appropriate to have an empirical insight into some specific questions. We have

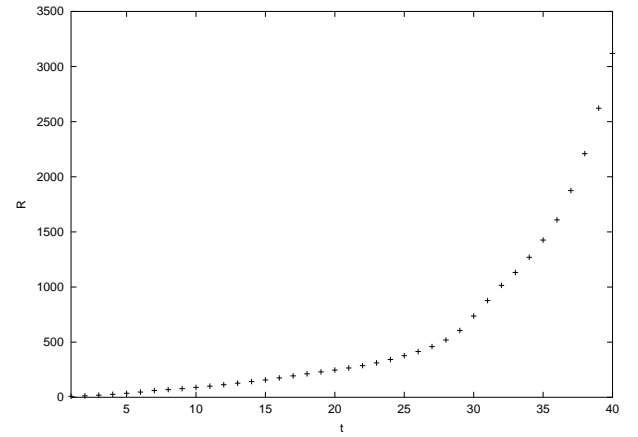


FIG. 3: The early growth of the cumulative capacity for revenue generation (spanning the first forty years of available data from *IBM*) shows an exponential pattern.

already indicated that our theory has been bolstered by a survey of the growth pattern of the multi-national company, *IBM*. This organization has been in existence in its presently known form for nearly a century. Besides this, it has spread all over the globe. So on both of these counts, a company like *IBM* is ideally suited for our study. Data about its annual revenue generation and human resource strength, dating from the year 1914, have been published on the company website. Both these indices of the health of the company have been useful for us. In Fig. 2 we have presented a logarithmic plot of how the revenue generating ability of *IBM* has grown *cumulatively* since the year 1954. The prescription of the cumulative growth needs to be stressed upon here. For any evolving system its rate of growth is almost always a direct mathematical function of its current state. Frequently this dependence leads to an exponential growth pattern (something that, as we shall presently demonstrate, is quite relevant in this case also). Mindful of this fact, we have transformed the annual revenue generated by *IBM* into a cumulative revenue index,  $R$ , which is measured in millions of dollars. This index has been plotted along the vertical axis of Fig. 2, with the horizontal axis measuring the logarithm of the time (scaled in years). The similarity of this plot with all the model solutions shown in Fig. 1 is very apparent. The most interesting feature is the diversion into a mode of reduced growth rate in the very last decade of *IBM*. This is exactly the terminal growth phase that we have depicted theoretically in Fig. 1.

All this similarity between the empirical data and our theoretical model is undoubtedly pleasing to note, but one could actually go much beyond this. The model given by Eq. (4) shows that the early power-law type of behaviour will start developing right from the inception of an organization. However, this is not exactly what the *IBM* data indicate. For forty years since 1914, the company has registered a growth pattern in its cumulative revenue that is visibly exponential, something that is quite evident from Fig. 3.

Taking the information contained in Figs. 2 & 3 together,

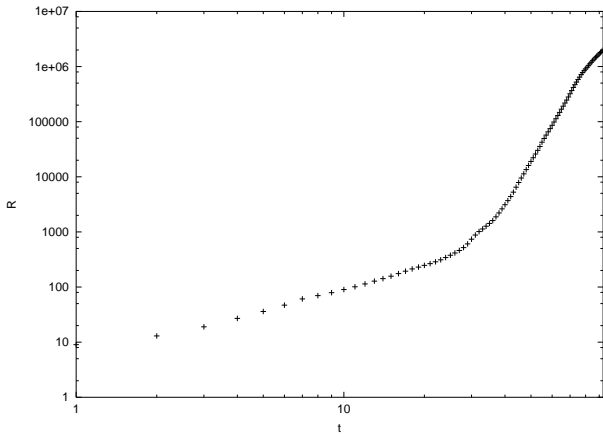


FIG. 4: A logarithmic plot of the cumulative revenue growth against time for nearly a century, shows that the early exponential growth of *IBM* has been saturated in the later stages.

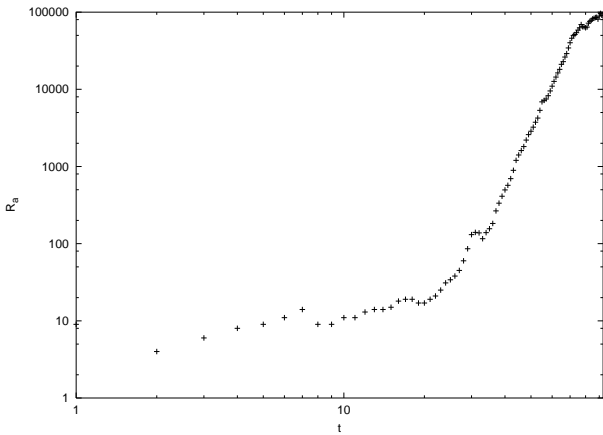


FIG. 5: A logarithmic plot of the annual revenue,  $R_a$ , against time. It shows some fluctuations from year to year, but the qualitative similarity with the trend indicated by Fig. 4 is evident. In fact the saturation towards a terminal state looks more prominent in this plot.

a clear impression of the overall fortunes of the company can be derived from Fig. 4, which gives a logarithmic plot of the cumulative revenue growth of *IBM* over nearly a century of its existence. For comparison this may also be viewed against the plot in Fig. 5, which shows the growth in the *annual* revenue generated (in millions of dollars) every year. The trend in the two plots are largely identical, except for some understandable fluctuations in the latter.

There is a very interesting aspect to both Figs. 4 & 5. Around the fortieth year of *IBM*, its growth seemed to have deviated noticeably from its early exponential rise. This could only have resulted from a stagnation that *IBM* might have faced in its native base. To overcome this impasse, it became necessary for *IBM* to extend its core competency levels, as well as undergo a concerted process of geographical expansion. And indeed in the decade following the Second

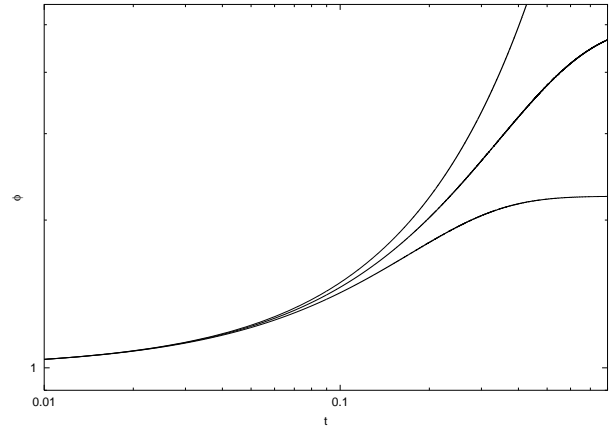


FIG. 6: A logarithmic plot of  $\phi$  against  $t$ , to show how early exponential growth can be saturated on later time scales by nonlinear factors. Under the initial condition,  $\phi = 1$  at  $t = 0$ , the three curves from the top to the bottom correspond to  $\alpha = 0, 1$  and  $2$ , respectively. Terminal features become more pronounced for higher values of  $\alpha$ . All three curves have been drawn for  $a = 5$  and  $b = 1$ .

World War, *IBM* applied itself assiduously to the innovation and development of increasingly newer technology (the details of which are available on the *IBM* website), and to expanding its operational base overseas (for instance, *IBM* Israel was founded in 1950 and *IBM* United Kingdom in 1951).

In the long run, however, even these measures proved insufficient in sustaining growth, and at present *IBM* has globally lapsed into the terminal phase. This observation is in perfect consonance with our earlier reasoning about how growth rates can become progressively diminished even after an organization employs a policy of expansion. First, it will be cramped for space once its very own size becomes comparable with the largest scales of available physical space, and secondly, various dissipative factors — all of them presumably nonlinear in nature — assume a very effective role to throttle any further growth.

Having taken cognizance of the inescapable empirical fact that after enjoying a period of early exponential rise (which implies a beginning with a finite non-zero value at  $t = 0$ , and then gradually capitalizing on this start), an industrial organization will have to ultimately suffer its growth being stifled, we are in consequence led to the conclusion that our relatively simple model, based on the assumption of an early linear growth, as described by Eq. (3), will not entirely suffice for our purposes of building an analytical model for terminal behaviour. So we will now have to posit a nonlinear mathematical model that will represent a significant improvement over the simple linear differential equation described by Eq. (3). According to this necessity, as a simple extension of the linear model, we put forward a nonlinear growth model, suited for all time scales as

$$\frac{d\phi}{dt} = \phi(a - b\phi^\alpha), \quad (5)$$

in which  $\alpha$  is an exponent, which, for the sake of simplicity,

is restricted to be either zero or a positive integer. We also impose the requirement that  $a > b$ , with both of these parameters being positive.

Since Eq. (5) is a first-order differential equation in time, we will need only one initial condition to fix the first integral of the equation. To this end we require that  $\phi = \phi_0$  at  $t = 0$ , for any value of  $\alpha$ . Let us first consider the case of  $\alpha = 0$ . This will immediately reduce Eq. (5) to a linear differential equation, whose integral solution will read as

$$\phi = \phi_0 e^{(a-b)t}. \quad (6)$$

This solution, of course, represents an unbridled exponential growth that can be sustained perpetually. The topmost curve in Fig. 6 depicts this behaviour.

More interesting properties will result from Eq. (5) when  $\alpha > 0$ . This case will imply that the exponential growth exhibited by Eq. (6) will be saturated by a nonlinear damping term. For  $\alpha = 1$ , the damping term will be of the second order of nonlinearity. The corresponding integral solution, obtained by the method of partial fractions, will be

$$\phi = \frac{ac}{bc + e^{-at}}, \quad (7)$$

in which  $c = \phi_0(a - b\phi_0)^{-1}$ . This solution is represented by the second curve from the top in Fig. 6. The profile of this theoretical solution has much closeness with the two empirical growth profiles in Figs. 4 & 5, respectively.

Prescribing  $\alpha = 2$  will raise the damping term in Eq. (5) to the third order of nonlinearity, and the integral solution will accordingly be obtained as

$$\phi = \frac{ck}{\sqrt{c^2 + e^{-2at}}}, \quad (8)$$

with  $c = \phi_0(k^2 - \phi_0^2)^{-1/2}$  and  $k^2 = a/b$ . This solution has been plotted as the lowermost curve in Fig. 6. What we can immediately conclude from the behaviour of the three curves in Fig. 6 is that with progressively higher orders of nonlinearity, the damping of the growth of  $\phi$  is saturated towards the terminal phase with greater promptness. And much more importantly for our study, we can also appreciate that the characteristic of nonlinear saturation of exponential growth is very much in keeping with what the *IBM* data indicate regarding growth in the revenue generating abilities of an industrial organization. When  $\alpha > 0$ , the terminal value of  $\phi$  will be given by  $(a/b)^{1/\alpha}$ , and the onset of nonlinear damping will occur when the magnitude of  $\phi$  becomes comparable with this terminal value. It is not difficult to see that near this critical value, any further growth of  $\phi$  would be inhibited by its own size. For the specific cases of Eqs. (7) & (8) this contention is easily verified for  $t \rightarrow \infty$ . In the opposite limit of  $t \rightarrow 0$ , it is equally apparent that both the damped solutions will merge into an exponential pattern.

Finally, we note that the value of  $\alpha$  has a bearing on the symmetry of Eq. (5). We have constrained  $\alpha$  to belong to the set of positive integers. When  $\alpha$  is an even number, Eq. (5) retains an invariant form under the transformation,  $\phi \rightarrow -\phi$ . On the other hand, when  $\alpha$  is an odd number, this invariance

breaks down. In physics, this argument, based on considerations of symmetry, is crucial for the precise choice of equations governing various nonlinear growth phenomena [2]. For the specific instances of  $\alpha = 1$  and  $\alpha = 2$  in Eq. (5), there are many analogous equations in nonlinear dynamics. To cite one such example with regard to the former case, there is the law of mass action of chemical kinetics [4]. With regard to the latter, there is the nonlinear saturation of the amplitude of a standing wave in super-critical flows, in the fluid dynamical problem of the hydraulic jump [11].

Our industrial case study may not need to invoke such intricate details in making a correct choice of a value for  $\alpha$ . All the relevant industrial variables in our study are always measurably positive quantities, and so in a qualitative sense any integral exponent (odd or even) should be effective to bring about terminal growth. The simplest case, of course, is when  $\alpha = 1$ . So in concluding this discussion we can now claim that industrial growth patterns can be described on early time scales by an exponential law, which is later driven towards a limiting value by various retarding factors, all of which are nonlinear in character.

### III. THE BALANCED SCORECARD : A DYNAMICAL SYSTEMS PERSPECTIVE

One crucial challenge that organizations face as they mature and go beyond a critical size, is how to balance their key stakeholder expectations. In this context it will be topical to mention the paradigm of alignment (on which we have already dwelt at some length in Section II), using the ‘‘Balanced Scorecard’’ methodology. This was first proposed by Kaplan & Norton [7], who maintain that organizations need to align the value propositions at the enterprise level, the business unit level and the support function level to create sustained organizational growth, in addition to balancing the financial and non-financial goals of the organization.

In point of fact, Kaplan & Norton define a radically new perspective to the concept of alignment, and they strongly advocate that alignment should be instituted as a continuous process, rather than as a one-time annual ritual of goal setting and performance appraisal. An organization is a dynamic entity in a dynamically evolving macroeconomic environment where change is continuously to be expected. Any strategy and its implementation must, therefore, evolve accordingly. Otherwise an organization, happily aligned at one point of time, will soon become misaligned. This is somewhat reminiscent of the onset and growth of disorder (measured by entropy) in a thermodynamical system that is left to its own means [12]. By the same token and as a corollary of this physical analogy, the proposition of Kaplan & Norton implies that keeping an organization on its sustained growth path, will necessitate the management of its process of alignment through active intervention.

Frequently, however, it happens that even a very regular monitoring of industrial growth by means of one particular index only, may actually convey no more than a partial notion of the true state of affairs. In such a situation a more bal-

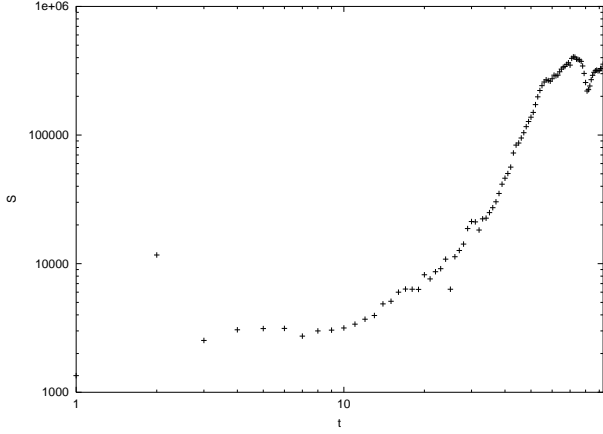


FIG. 7: A logarithmic plot of the growth of human resource content,  $S$ , against  $t$ . The growth indicated in this plot is compatible with the growth of  $R$  and  $R_a$ , as shown in Figs. 4 & 5, respectively. This implies that the growth of  $R$  and  $S$  are coupled to each other and are correlated.

anced view could be obtained by accounting for the behaviour of some other relevant variables. Indeed, the concept of the “Balanced Scorecard” is premised on this very principle [6]. Therefore, charting the individual properties of various key variables, and then analyzing them in conjunction with one another, can enable us to build a more comprehensive picture. The variables involved can be diverse in nature, ranging from the finances of an organization to its human and technological resources and to the market within which the organization might be operating. The growth rate of any one of these variables may have a correlated functional dependence on the current state of all the pertinent variables.

So if we are to study an industrial organization whose current state is defined completely by a set of  $n$  variables, then the growth rate of the  $i$ -th. variable,  $\phi_i$ , can be described as,

$$\frac{d\phi_i}{dt} = \Phi_j(\{\phi_i\}), \quad (9)$$

with  $\Phi_j$  being a general function of all the variables in the set. The whole set can be expressed explicitly by making both  $i$  and  $j$  run from 1 to  $n$ . In this situation we will have a set of  $n$  first-order differential equations, with each one of them being coupled to all the others. This entire set will form a first-order dynamical system, and for the sake of simplicity we require this dynamical system to be autonomous [4, 5].

We have already carried out an extensive study of the growth of *IBM*, by following the revenue (both cumulative and annual) that the company has been generating over the years. Alongside this, we would now like to study the behaviour of another variable — the human resource content,  $S$ . Its growth against time has been shown in Fig. 7. It is easy to discern that the general qualitative pattern of the growth of  $S$  has been in tandem with the growth of  $R$  or  $R_a$ , as Figs. 4 & 5 show them. Therefore, it stands to reason that the growth of both  $R$  (we prefer this variable to  $R_a$ ) and  $S$  are dynamically coupled to

each other in the form

$$\begin{aligned} \frac{dR}{dt} &= \rho(R, S) \\ \frac{dS}{dt} &= \sigma(R, S). \end{aligned} \quad (10)$$

The two foregoing expressions should occasion no mystery. If an industrial organization generates enough revenue, it should become financially viable for it to maintain a sizeable human resource pool. On the other hand, for a healthy organization, the human resource strength will translate into a greater ability to generate revenue. And in this manner both the revenue and the human resource content of an organization will sustain the growth of each other. The coupled set involving  $R$  and  $S$  in Eqs. (10) simply states this fact in mathematical terms.

The equilibrium condition is obtained when both the derivatives on the left hand side of Eqs. (10) vanish simultaneously, i.e.  $dR/dt = dS/dt = 0$ . The corresponding coordinates in the  $S$ — $R$  plane may be labelled  $(S_c, R_c)$ . Since the terminal state implies the cessation of all growth in time (i.e. all derivatives with respect to time will vanish), we are now in a position to argue that the equilibrium state in the  $S$ — $R$  plane actually represents a terminal state in real time growth.

Some general deductions can now be made about the nature of the equilibrium state, with the help of dynamical systems theory [4, 5]. The two coupled equations, given by Eqs. (10), will, in the most general sense, be nonlinear. A linearization treatment on them could be carried out by applying small perturbations on  $R$  and  $S$  about their equilibrium state values. The perturbation scheme will be  $R = R_c + R'$  and  $S = S_c + S'$ . This will allow us to express a coupled set of linearized equations as

$$\begin{aligned} \frac{dR'}{dt} &= \mathcal{A}R' + \mathcal{B}S' \\ \frac{dS'}{dt} &= \mathcal{C}R' + \mathcal{D}S', \end{aligned} \quad (11)$$

in which

$$\mathcal{A} = \left. \frac{\partial \rho}{\partial R} \right|_{R_c}, \quad \mathcal{B} = \left. \frac{\partial \rho}{\partial S} \right|_{S_c}, \quad \mathcal{C} = \left. \frac{\partial \sigma}{\partial R} \right|_{R_c}, \quad \mathcal{D} = \left. \frac{\partial \sigma}{\partial S} \right|_{S_c}.$$

Solutions of the form  $R' \sim e^{\lambda t}$  and  $S' \sim e^{\lambda t}$  will enable us to derive the eigenvalues,  $\lambda$ , of the stability matrix implied by Eqs. (11), as

$$\lambda = \frac{1}{2} \left[ (\mathcal{A} + \mathcal{D}) \pm \sqrt{(\mathcal{A} + \mathcal{D})^2 - 4(\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C})} \right]. \quad (12)$$

The exact determination of the values of the two roots of  $\lambda$  will impart a clear idea about the nature of the equilibrium state, which can either be a saddle point or a node or a focus [4, 5]. The last case will necessarily mean an oscillatory nature in the growth of both  $R$  and  $S$  [4, 5]. If we go back to Figs. 4 & 7, we will notice that the respective growth patterns of *both*  $R$  and  $S$  have, on the other hand, been largely monotonic in nature (except for a noticeable fluctuation in  $S$  at large values of  $t$ ). So an immediate conclusion that follows is that

the equilibrium state is very likely an unstable node [4, 5], and this will correspond mathematically to  $\lambda$  having two real positive roots. Practically speaking, this is what we should expect entirely. For an industrial organization it is not conceivable that while there is growth in one variable, there will be decay in the other. Both will have to grow in close mutual association, and, if our dynamical systems argument is anything to go by, both will make the approach towards the terminal state simultaneously. One variable cannot behave completely independently of the other.

There is, however, a problem inherent in all this. Since both the equilibrium states in  $R$  and  $S$  (which are also the terminal states of these two variables) represent a maximum possible growth in time, the whole configuration is unstable. Slight fluctuations can bring about radical and, very likely, unwelcome changes in this configuration. In this situation, a conventional approach to achieving high-growth performance (as, for instance, by arbitrarily increasing or reducing the human resource strength) will not only be merely cosmetic, but can actually be counter-productive. A truly effective remedy should be founded on a more imaginative and innovative approach.

#### IV. BECALMED IN THE “BLUE OCEAN” : DYNAMIC EVOLUTION TOWARDS THE TERMINAL STATE

It will now be both very relevant and very important to bring in here the concept of the “Blue Ocean Strategy”, introduced and popularized by Kim & Mauborgne [8], who, for the very first time, posited a contrary approach to strategy development. They colourfully refer to the traditional process of strategic planning as the “Red Ocean”, which, being based on competition, will inevitably lead to the shedding of “blood”. In a “Red Ocean” paradigm, a traditional organization focuses on competing against rivals and outmanoeuvring them. However, with supply exceeding demand in matured industries and markets, competing successfully for a share of contracting market spaces will not be sufficient to sustain high-growth performance.

In a “Blue Ocean” strategic paradigm of growth, on the other hand, organizations concentrate on “value innovation” — doing something beyond the conventional approach and not using competition as the benchmark. Instead the focus is on making competition irrelevant by creating a leap in value for customers and the organization, and thereby continuing to open up new and uncontested market spaces. This makes it apparently self-evident that if organizations successfully continue to create their niches of “Blue Ocean”, they may extend their period of exponential growth or defer their phase of terminal growth. We make an attempt to address these issues now through our particular approach.

First let us consider the “Red Ocean” scenario. In doing so it will be instructive for us to invoke an analogy where the possibility of bloodshed, in its truest sense, is very much real — that of two species of animals, with population sizes,  $\phi_1$  and  $\phi_2$ , competing for the control of a common food resource. This condition can be expressed in terms of a coupled

autonomous nonlinear dynamical system as

$$\begin{aligned}\frac{d\phi_1}{dt} &= [a_1 - d_1(b\phi_1 + c\phi_2)]\phi_1 \\ \frac{d\phi_2}{dt} &= [a_2 - d_2(b\phi_1 + c\phi_2)]\phi_2,\end{aligned}\quad (13)$$

from which it is possible to show that if  $a_1d_2 > a_2d_1$ , then the population size given by  $\phi_2$  dies out, while the population size,  $\phi_1$ , approaches a limiting value [5]. This is one manifestation of Volterra’s Exclusion Principle [5]. Applied to a situation where there is competition between two (or more) rival organizations for the control of a common market space, even the wresting of monopolistic control by any one organization will not ensure its high-growth performance in the long run. So there will be no protracted savouring of the fruits of absolute victory over all rivals in the “Red Ocean”. This, of course, provides robust justification for adopting the “Blue Ocean Strategy”. However, we shall now argue that even in doing so, one has to face the stark reality of growth being terminated ultimately.

By its very definition the “Blue Ocean” conjures up an image of vast openness. But does this actually imply an infinitely available space for growth? In Section II we presented some models in which growth was being retarded by various dissipative factors, either linear, or, more realistically, nonlinear. In a discussion on what brings growth to a halt, we alluded to the finite size of a system being one of the reasons that can constrain growth. This is something that we shall take up in greater detail here, with the help of some pedagogical models.

Let us say that the “Blue Ocean” defines the maximum available space in a niche that an organization has defined for its own growth. As it was before, the state of the organization is indicated by the variable,  $\phi$ . However, unlike what we did before, requiring  $\phi$  to be dependent on  $t$  only, this variable will now also be a function of the spatial dimension,  $x$ , that it occupies inside the “Blue Ocean” (which, for simplicity, we treat as one-dimensional). To enunciate this mathematically, we will write  $\phi \equiv \phi(x, t)$ . Let us also assume that none of the usual retarding factors is operating against the growth of  $\phi$ , i.e. the system is a conservative one.

Considering first a linear model of growth through the “Blue Ocean”, we take up the diffusion equation,

$$\frac{\partial\phi}{\partial t} = \Delta \frac{\partial^2\phi}{\partial x^2}.\quad (14)$$

Under the initial condition,  $\phi(x, 0) = \delta(x)$ , the Dirac delta function, we can arrive at the solution,

$$\phi(x, t) = \frac{1}{\sqrt{4\pi\Delta t}} \exp\left(-\frac{x^2}{4\Delta t}\right).\quad (15)$$

This is the “point source solution” of the diffusion equation [13]. It means that if  $\phi$  were to start initially with an abrupt and high spurt at a given point, it would subsequently diffuse (through the “Blue Ocean”, as it were) according to the Gaussian distribution law given by Eq. (15). What emerges is that left to its own devices, there will be no growth whatsoever



for  $\phi$ , but only its global redistribution, as the diffusion process progresses gradually. This is hardly surprising, because the diffusion model is a linear model, with nothing to drive the growth of  $\phi$ .

However, this is not to say that no growth could ever be described by a linear equation. For instance, the addition of a random noise term, dependent on both  $x$  and  $t$ , on the right hand side of Eq. (14) will alter this equation to the form of the Edwards-Wilkinson equation [2, 14], which is a linear equation that is regularly invoked to follow the growth of an interface by the random deposition of particles with surface relaxation [2]. As a result of this relaxation, a correlated surface growth will be obtained through the redistribution of matter at the interface, after starting from flat initial conditions.

Inclusion of nonlinearity, on the other hand, generalizes the Edwards-Wilkinson equation to the Kardar-Parisi-Zhang equation [2, 15]. The role of nonlinearity is crucial in the latter case. It generates the growth of an interface by *adding* material, as opposed to *reorganizing* the interface through redistribution of matter. So this is a situation where nonlinear effects play a more influential role in growth. Having said this, we must also point out that in keeping with the principle of our work, growth models described both by the Edwards-Wilkinson and the Kardar-Parisi-Zhang equations come to a saturated end at a limiting scale of length [2].

With regard to our specific objective of understanding how nonlinear effects may improve on a linearized model, we now devise a simple first-order nonlinear mathematical model by which the growth of the field,  $\phi(x, t)$ , will be described. We write it as

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = -V'(x). \quad (16)$$

The foregoing equation bears the form of pressure-free Euler's equation [9], which finds its use in the study of astrophysical gas dynamics [16]. The term on the right hand side gives the force that drives the growth of  $\phi$ . This force is expressed as the negative gradient (the prime refers to a derivative with respect to  $x$ ) of a potential function,  $V(x)$ .

A solution of Eq. (16) can be obtained by the method of characteristics [17], which will entail our writing

$$\frac{dt}{1} = \frac{dx}{\phi} = \frac{d\phi}{-V'(x)}. \quad (17)$$

The solution of the static  $d\phi/dx$  equation is easy to find. It is

$$\frac{\phi^2}{2} + V(x) = \frac{c^2}{2}, \quad (18)$$

where  $c$  is an integration constant, which, for this integration, can in general be a function of  $t$ . It shall be presently demonstrated that it actually vanishes terminally in time. The result of Eq. (18) can be used to find the solution of the  $dx/dt$  equation. We can write this as

$$t = \pm \int \frac{dx}{\sqrt{c^2 - 2V(x)}}, \quad (19)$$

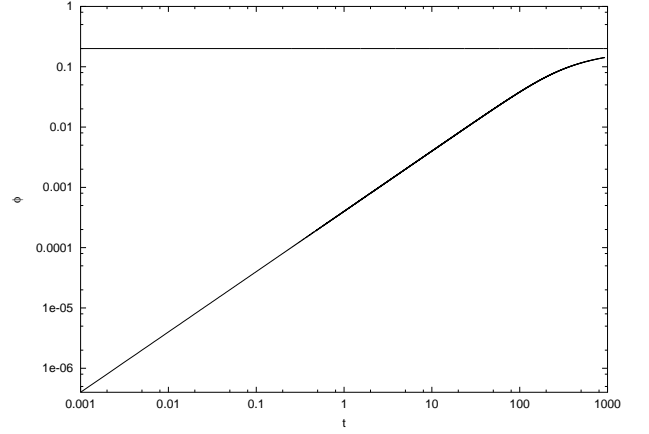


FIG. 8: A logarithmic plot of the temporal evolution of  $\phi$  through  $t$ , at a fixed value of  $x = 50$ . The slope of this logarithmic plot shows that the early stages of the evolution follow a linear power law. Deviation from this trend sets in later. The horizontal line at the top shows the terminal value that  $\phi$  may attain as  $t \rightarrow \infty$ .

from which, for our subsequent analysis, we will choose the positive sign of the square root by the self-evident physical criterion that  $t > 0$ .

Prescribing a functional form of  $V(x)$  will now be necessary. We have some general expectations about the driving force. It will enable our system to stay well-knit, but at the same time will allow it to expand gently. This is certainly to be much preferred to a violently fissiparous outward disintegration (which assuredly is never a desirable mode of industrial expansion). In a very large “Blue Ocean”, we should also expect the influence of the driving force to vanish at the outer boundary, from a simple requirement of well-behaved boundary conditions. These criteria can be mathematically fulfilled by various prescriptions for  $V(x)$ , but here we choose a very familiar and most elementary expression, that of the Newtonian potential,  $V(x) = -1/x$ . The negative sign in the potential keeps our system self-contained, while the inverse law satisfies our required boundary condition. In studies of astrophysical accretion, the “pressure-free” spherically symmetric infall of gas has been studied in great detail by using this potential [18, 19].

With the Newtonian potential defined for  $V(x)$ , it will now become possible to derive an exact analytical solution for Eq. (19) as

$$c\phi x - \ln \left[ x \left( \frac{\phi}{c} + 1 \right)^2 \right] - c^3 t = \tilde{c}. \quad (20)$$

The general solution of Eq. (16) will be delivered under the condition  $f(\tilde{c}) = c^2/2$ , with the functional form of  $f$  having to be determined by the initial condition under which an organization begins its voyage in the “Blue Ocean”. Realistically speaking, one suitable initial condition can be the globally flat condition,  $\phi(x, 0) = 0$ . At subsequent times the evolution of  $\phi$  will be driven by our chosen force, and the resultant analyt-

ical relation for the growth of  $\phi$  will be given by

$$\phi^2 - \frac{2}{x} \left[ 1 - \left( \frac{\phi}{c} + 1 \right)^{-2} \exp(c\phi x - c^3 t) \right] = 0. \quad (21)$$

It can now be seen that for  $t \rightarrow \infty$ , we will have the solution,

$$\phi^2 = \frac{2}{x}, \quad (22)$$

i.e. through an infinite passage of time,  $\phi$  can only grow up to a limiting value. This limit will be determined by the spatial dimensions within which an organization will be operating. We make a graphical representation of our arguments in Fig. 8 by applying finite differencing techniques on Eq. (16). It can be seen in this plot that at a given spatial position, the temporal evolution of the  $\phi$  field will reach a terminal value after an infinite elapse of time. This property will be independent of the chosen spatial scale.

If, suppose, we choose another initial condition,  $\phi(x, 0) = \phi_0$ , which is a constant non-zero initial value of  $\phi$ , then we can show that for  $t \rightarrow \infty$ , we will have  $\phi$  converging according to,

$$\phi^2 = \phi_0^2 + \frac{2}{x}, \quad (23)$$

which underscores once again how a limit on the growth of  $\phi$  can be set by a local length scale.

So after this analysis we can make a positive claim that we have provided an illustrative mathematical model of how in the most optimistic possible scenario (the “Blue Ocean”), where there is no dissipative factor to contend with, the growth of an industrial organization will still become terminal in nature, simply because the environment within which the organization operates is conservative and has practical limitations imposed upon it by finite but well-behaved boundary conditions.

## V. UNIVERSAL FEATURES AND PREDICTIONS

Our objective in undertaking this entire study has primarily been to provide a mathematically viable basis for our argument that just like any natural physical system, an industrial organization will also have an end to its growth. In doing so, we chose the growth of *IBM* as our case in point. We have also dwelt at length on what had prompted our choice, namely, the fact that *IBM* being a player on the global stage, its growth figures would be free of local fluctuations. Besides, *IBM* has been in existence for nearly a century, and, therefore, it is ideally suited as a case study over long time scales. While it has been gratifying for us to note that we have come beguilingly close to furnishing an accurate analytical model for industrial growth, it has also broached a very serious question: What can be done to defer the onset of the terminal phase in the life of an organization? Resignation in the face of the inevitable cannot be an answer. One might as well argue that since death awaits all mortals, extending the life-expectancy of individuals through the advancement of medical science should be a colossally futile exercise.

Let us, therefore, first diagnose the general symptoms of the terminal phase in our mathematical model. The parameter,  $b$ , in both Eqs. (3) & (5), has been identified as being connected to saturation in growth. We have also argued, with the help of the *IBM* data, that Eq. (5) gives a much better global model for terminal behaviour than Eq. (3). This will naturally suggest that the retarding factors acting against the growth of an organization are nonlinear in character. These factors are all quantified through the parameter  $\alpha$  in Eq. (5). We have already shown in Figs. 1 & 6 that reducing  $b$  and  $\alpha$ , respectively, will be conducive to lasting growth. So, taken together, a successful management strategy for feasible long-term growth will entail tuning both  $b$  and  $\alpha$  periodically (alignment monitoring, so to speak) to reduce their negative influence. The “Blue Ocean” strategy is certainly a practical and effective step in that direction, but our mathematical modelling cautions us against placing our absolute trust in it.

A more comprehensive approach should necessitate making precise numerical estimates in relation to the various retarding parameters in our model. This can only be done empirically and on a sufficiently extensive scale. The rewards of this pain-staking exercise, however, could be far-reaching as well as revealing. As an example of this method, we found that the growth curve of *IBM* started straggling noticeably behind a purely exponential growth pattern around the fortieth year of the company, since its inception under its presently well-recognized brand name. While this indicated the dissipative influence of nonlinear factors in a mathematical sense, from a more practical point of view in identifying these factors, it also pointed to a possible connection between the slowing down in growth and a global expansion overseas. We believe that much more useful information could be gleaned if similar analytical methods were to be applied to study the growth behaviour of other industrial organizations. This could enable us to identify the common denominators active behind terminal growth.

All the same, a precise diagnosis of the general malaise will only amount to addressing a part of the question of how to defer the onset of the terminal phase. The more important part would be to define and present a solution. Admittedly, it will never be an easy task to devise an effective strategy, based on quantitative measures, to sustain a steady momentum in industrial growth. It will be even more difficult to make such a strategy successfully applicable in all cases.

To elucidate the last point further, let us consider the background of *IBM*, on which our case study has been based. It is a company that has its primary base in the US, and is a key player in the global computer industry. In fact *IBM* is important enough to be a representative organization of the industrial domain to which it belongs. In consequence of this fact, we may entertain the view that in its general import at least, our model for the growth of *IBM* can also be extended to make a preliminary foray into studying the growth of other representative organizations from different industries. Nevertheless, in doing so we may encounter great variations in matters related to the details. Just as animal species thrive and evolve in their own ecological niches, so is it with industries. It is largely true that each one of them thrives in its own chosen and well-defined domain, catering to the specific de-

mands of a particular market, and deriving their sustenance from the specialized competency of a trained workforce. As a result different kinds of industries may pose its own specific array of complex problems, which could only be understood on a case-by-case basis. While the controlling factors in all these cases may be economic in essence, it is also very much possible that socio-political circumstances may leave their imprint on the fortunes of any industry. It is certainly difficult to deny that the social, political and economic framework of the country of origin of any industrial organization will all collectively have a decisive role to play in determining the early extent of the growth of the organization. For instance, it is hardly to be expected that a representative company of the Japanese automobiles industry will show exactly the same features as, let us say, a German pharmaceuticals company. Indeed, two different industries from the same country could also very likely display different traits. We are strongly of the view that all these aspects of an organization and its history will manifest themselves in measurable terms through various growth-related parameters. Therefore, a study of these parameters is of paramount importance in determining what the nature of the solution should be to bring about sustained growth.

This will also be quite useful for organizations that are still in the early stages of their growth. Armed with a knowledge of the general nature of the possible difficulties that lie ahead, a company can always apply an appropriately corrective measure at the right juncture. As a result they will be capable of making a more effective implementation of future strategies and innovative solutions for growth, all of which should ideally be in a state of adaptable alignment with their objectives

and core competency.

On the other hand, companies which would already have entered a terminal phase, might also draw some object lessons from these studies, in the form of proper reality checks. This will allow for a more functional and timely redefining of fundamental objectives. The solutions which might follow, could be varied in many unexpected ways. Rather than adhering to conventional modes of growth and survival, industries could devise ways of preserving both their existence and their relevance by being more integrated with the social welfare of their markets — in short, assist (perhaps even, in a partial measure at least, assume roles which hitherto were largely the preserve of the state) in creating an environment of overall prosperity, and a general feeling of well-being, right alongside the generation of wealth. The possibilities of deriving direct and indirect benefits through such inclusive and enlightened strategy implementation and objectives are myriad. Their social, political and, certainly, economic impact will be both lasting and salutary. We propose to take up these and many other related issues in subsequent studies.

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